Mathematical Finance
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Assignment 9

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

Asian option in Black–Scholes model

We consider the one-dimensional Black–Scholes model seen in class, with time horizon $T > 0$, such that the unique risky asset in the market has the following dynamics under the unique risk-neutral measure $\mathbb{Q}$

$$S_t = S_0 + \int_0^t rS_s \mathrm{d}s + \int_0^t \sigma S_s \mathrm{d}B_s^{\mathbb{Q}},\ t \in [0, T],$$

where $r \geq 0$ is the interest-rate, $\sigma > 0$ the volatility of S, $S_0 > 0$ its initial value, and $B^{\mathbb{Q}}$ is an $(\mathbb{F}, \mathbb{Q})$–Brownian motion.

The goal of the exercise is to evaluate the price of an Asian Call option on S, with maturity T and strike $K > 0$, whose payoff at maturity is given by

$$\Phi_T := \left(\frac{1}{T}\int_0^T S_s \mathrm{d}s - K\right)^+.$$

We denote by $\mathrm{AC}_t(T, K; S)$ the value at any time $t \in [0, T]$ of such an option, and as usual, by $C_t(T, K; S)$ the value at time $t \in [0, T]$ of the call option with strike K, maturity T and underlying S.

1) Consider another option with maturity T, whose payoff is defined by

$$\Psi_T := \left(\exp\left(\frac{1}{T}\int_0^T \log(S_s)\mathrm{d}s\right) - K\right)^+$$

The value at any time $t \in [0, T]$ of the option with payoff Ψ_T is denoted by $\mathrm{GAC}_t(T, K; S)$.

a) Prove that

$$\Phi_T = \left(\frac{1}{T}\int_0^T (S_s - K)\mathrm{d}s\right)^+,$$

and then that

$$\Phi_T \leq \frac{1}{T}\int_0^T (S_s - K)^+ \mathrm{d}s.$$

b) Prove that

$$\exp\left(\frac{1}{T}\int_0^T \log(S_s)\mathrm{d}s\right) \leq \frac{1}{T}\int_0^T S_s \mathrm{d}s,$$

and deduce that

$$\Psi_T \leq \Phi_T.$$

Hint: It would be profitable to admit the so-called Jensen's inequality, which states that for any convex function $f : \mathbb{R} \longrightarrow \mathbb{R}$, *and any integrable function* $g : [0, T] \longrightarrow \mathbb{R}$, *we have*

$$f\left(\frac{1}{T}\int_0^T g(s)\mathrm{d}s\right) \leq \frac{1}{T}\int_0^T f\big(g(s)\big)\mathrm{d}s.$$

c) Deduce then that

$$\mathrm{GAC}_0(T, K; S) \leq \mathrm{AC}_0(T, K; S) \leq \frac{1}{T}\int_0^T \mathrm{e}^{-r(T-s)}\mathrm{C}_0(u, K; S)\mathrm{d}u.$$

2)a) Show that

$$\int_0^T B_s^{\mathbb{Q}}\mathrm{d}s = \int_0^T (T-s)\mathrm{d}B_s^{\mathbb{Q}}.$$

2)b) Deduce that the random variable $1/T\int_0^T \log(S_s)\mathrm{d}s$ has a Gaussian distribution under $\mathbb{Q}$, with mean m and variance v^2, where

$$m = \log(S_0) + \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2},\ v^2 := \frac{\sigma^2 T}{3}.$$

2)c) Show then (this should remind you of Black–Scholes formula) that

$$\mathrm{GAC}_0(T,K;S) = S_0\mathrm{e}^{-\frac{T}{2}(r+\frac{\sigma^2}{6})}\mathcal{N}(d_1) - K\mathrm{e}^{-rT}\mathcal{N}(d_0),$$

where

$$d_0 := \frac{1}{\sigma\sqrt{T/3}}\log\left(\frac{S_0\mathrm{e}^{\frac{T}{2}(r-\frac{\sigma^2}{2})}}{K}\right),\ d_1 := d_0 + v.$$

3) We now define a new process

$$Y_t = Y_0 + \int_0^t S_u\mathrm{d}u,\ t\in[0,T],$$

where $Y_0 > 0$ is a given constant.

a) Prove that the pair (S,Y) is a Markovian diffusion, that is to say that you can find maps $b : (0,+\infty)^2 \longrightarrow \mathbb{R}^2$ and $\Sigma : (0,+\infty)^2 \longrightarrow \mathbb{R}^{2\times 2}$ (where $\mathbb{R}^{2\times 2}$ is the set of 2×2 matrices) such that

$$\begin{pmatrix} S_t \\ Y_t \end{pmatrix} = \begin{pmatrix} S_0 \\ Y_0 \end{pmatrix} + \int_0^t b(S_s,Y_s)\mathrm{d}s + \int_0^t \Sigma(S_s,Y_s)\begin{pmatrix} \mathrm{d}B_s^{\mathbb{Q}} \\ \mathrm{d}B_s^{\mathbb{Q}} \end{pmatrix},\ t\in[0,T].$$

b) We consider an option with maturity T and payoff $(1/TY_T - K)^+$. We are looking for a self-financing portfolio $X^{x,\Delta}$ which replicates this option and takes a very specific form, namely

$$X_t^{x,\Delta} = u(t,S_t,Y_t),\ t\in[0,T],$$

where the map $u : [0,T]\times(0,+\infty)^2 \longrightarrow \mathbb{R}$ is supposed to be as smooth as necessary. Using the two-dimensional Itô's formula show that necessarily, the map u must then satisfy the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x,y) + rx\dfrac{\partial u}{\partial x}(t,x,y) + x\dfrac{\partial u}{\partial y}(t,x,y) + \dfrac{\sigma^2}{2}x^2\dfrac{\partial^2 u}{\partial x^2}(t,x,y) - ru(t,x,y) = 0,\ (t,x,y)\in[0,T)\times(0,+\infty)^2, \\ u(T,x,y) = \left(\dfrac{y}{T} - K\right)^+,\ (x,y)\in(0,+\infty)^2. \end{cases}$$

We will assume that this PDE has a unique non-negative solution with bounded derivatives with respect to x and y.

c) Deduce a replicating strategy for the option with payoff Φ_T, and prove as well that

$$\mathrm{AC}_0(T,K;S) = u(0,S_0,0).$$

4) The PDE derived in the previous question is not easy to deal with numerically, so we will try to reduce its dimension.

a)($\star$) Prove that an option with maturity T and payoff $1/T\int_0^T S_s\mathrm{d}s - K$ can be replicated by a self-financing portfolio $(x^\star,\Delta^\star)$, where

$$x^\star := S_0\frac{1-\mathrm{e}^{-rT}}{rT} - K\mathrm{e}^{-rT},\ \Delta_t^\star := \frac{1-\mathrm{e}^{-r(T-t)}}{rT},\ t\in[0,T].$$

b) We define $Z_t := X_t^{x^\star,\Delta^\star}S_t^{-1}$, $t\in[0,T]$. Show that

$$\mathrm{d}Z_t = \big(\Delta_t^\star - Z_t\big)\sigma\big(\mathrm{d}B_t^{\mathbb{Q}} - \sigma\mathrm{d}t\big).$$

c) Let us consider the probability measure $\mathbb{Q}^S$ whose density with respect to $\mathbb{Q}$ is given by

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}} := \mathrm{e}^{-rT}\frac{S_T}{S_0}.$$

Show that $\mathbb{Q}^S$ is well-defined, that the process B^S, defined by $B^S_t := B^{\mathbb{Q}}_t - \sigma t$, $t \in [0,T]$, is an $(\mathbb{F}, \mathbb{Q}^S)$–Brownian motion, and deduce that Z is an $(\mathbb{F}, \mathbb{Q}^S)$-martingale.

d) Show that

$$\mathrm{AC}_0(T,K;S) = S_0\mathbb{E}^{\mathbb{Q}^S}[Z_T^+],$$

and deduce then that $\mathrm{AC}_0(T,K;S) = S_0 V(0,Z_0)$, where the function V is assumed to be smooth and solves the PDE

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,z) + \dfrac{\sigma^2}{2}\big(\Delta^\star_t - z\big)^2\dfrac{\partial^2 V}{\partial z^2}(t,z) = 0,\ (t,z) \in [0,T) \times (0,+\infty), \\ V(T,z) = z^+,\ z \in (0,+\infty). \end{cases}$$

Asian option on the arithmetic mean

Fix some horizon $T > 0$. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where $\mathbb{Q}$ is directly assumed to be a risk-neutral measure, under which the dynamics of the unique risky asset S in the market is given by

$$S_t = S_0 + \int_0^t rS_s\mathrm{d}s + \int_0^t S_s\sigma_s\mathrm{d}B^{\mathbb{Q}}_s,\ t \geq 0,$$

where $S_0 > 0$, where $B^{\mathbb{Q}}$ is an $(\mathbb{F},\mathbb{Q})$−Brownian motion, where $r \geq 0$ is the (constant) short rate, and where the process $(\sigma_t)_{t\geq 0}$ is an $\mathbb{F}$−adapted process satisfying $\underline{\sigma} \leq \sigma_t \leq \overline{\sigma}$, $t \geq 0$, for some constants $0 < \underline{\sigma} \leq \overline{\sigma} < +\infty$. We are interested in the pricing and hedging of the Asian option whose payoff G at maturity T is defined by

$$G := (Y_T - K)^+,\ \text{where } Y_t := \int_0^t S_u g(u)\mathrm{d}u,\ t \geq 0,$$

where $g : [0,+\infty) \longrightarrow \mathbb{R}$ is a given deterministic function, and the strike $K > 0$ is fixed.

1) Explain why we have

$$S_s = S_t\exp\bigg(\int_t^s\Big(r - \frac{\sigma_u^2}{2}\Big)\mathrm{d}u + \int_t^s \sigma_u\mathrm{d}B^{\mathbb{Q}}_u\bigg),\ \text{and } \mathbb{E}^{\mathbb{Q}}\bigg[\exp\bigg(-\int_t^s\frac{\sigma_u^2}{2}\mathrm{d}u + \int_t^s\sigma_u\mathrm{d}B^{\mathbb{Q}}_u\bigg)\bigg|\mathcal{F}_t\bigg] = 1,\ 0 \leq t \leq s.$$

2) Check then that you can write for any $t \in [0,T]$

$$\mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-r(T-t)}Y_T\big|\mathcal{F}_t\big] = \widehat{\theta}(t)S_t + \mathrm{e}^{-r(T-t)}\int_0^t g(u)S_u\mathrm{d}u,$$

where the deterministic function $\widehat{\theta} : [0,+\infty) \longrightarrow \mathbb{R}$ is defined by $\widehat{\theta}(t) := \int_t^T \mathrm{e}^{-r(T-u)}g(u)\mathrm{d}u,\ t \geq 0.$

3) Consider the self-financing portfolio with initial capital 0 and which consists in holding exactly $\widehat{\theta}(t)$ units of S at any $t \in [0,T]$. We denote the corresponding value by $X^{0,\widehat{\theta}}$. Show that

$$\mathrm{e}^{-rt}X_t^{0,\widehat{\theta}} = \int_0^t \widehat{\theta}(s)\sigma_s\mathrm{e}^{-rs}S_s\mathrm{d}B^{\mathbb{Q}}_s,\ t \in [0,T].$$

4) Denoting as usual $\widetilde{S}_t := \mathrm{e}^{-rt}S_t,\ t \geq 0$, the discounted value of the risky asset, show that

$$\widehat{\theta}(0)S_0 + \int_0^T \widehat{\theta}(u)\mathrm{d}\widetilde{S}_u = \mathrm{e}^{-rT}Y_T.$$

What does the left-hand side represent? Comment.

5) Deduce from the previous questions that

$$Y_T - K = X_T^{x_\circ,\widehat{\theta}},\ \text{where } x_\circ := \widehat{\theta}(0)S_0 - \mathrm{e}^{-rT}K.$$

6) Justify that the probability measure $\widetilde{\mathbb{Q}}$, whose density with respect to $\mathbb{Q}$ on $\mathcal{F}_t$, is given by $Z_t := \frac{\widetilde{S}_t}{S_0},\ t \in [0,T]$, is well defined.

7) Define now for simplicity the quantity $\xi_t := X_t^{x_\circ,\widehat{\theta}}/S_t,\ t \in [0,T]$. Show that we have

$$\mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-rT}G] = S_0\mathbb{E}^{\widetilde{\mathbb{Q}}}[\xi_T^+].$$

8) Explain why the process $\big(B_t^{\widetilde{\mathbb{Q}}}\big)_{t\in[0,T]}$ is an $(\mathbb{F},\widetilde{\mathbb{Q}})-$Brownian motion, where

$$B_t^{\widetilde{\mathbb{Q}}} := B_t - \int_0^t \sigma_s\mathrm{d}s,\ t \in [0,T].$$

9) Show using Itô's formula that

$$\xi_t = \frac{x_\circ}{S_0} + \int_0^t \big(\widehat{\theta}(s) - \xi_s\big)\sigma_s\mathrm{d}B_s^{\widetilde{\mathbb{Q}}},\ t \in [0,T].$$

10)($\star$) Assume from now on (that is until the end of the current exercise) that the process $(\sigma_t)_{t\in[0,T]}$ is constant equal to some $\sigma > 0$. Using the previous question, argue that there exists a function $v : [0,T] \times (0,+\infty) \longrightarrow \mathbb{R}$ such that

$$\mathbb{E}^{\widetilde{\mathbb{Q}}}[\xi_T^+|\mathcal{F}_t] = v(t,\xi_t),\ t \in [0,T].$$

Assuming that v is smooth, give the PDE (partial differential equation) satisfied by v, and deduce the price at time 0 of the Asian option with payoff G in terms of v, and S_0.

11)($\star$) We now impose in addition that $g(t) = 1/T,\ t \geq 0$. Define for simplicity $k := -\log(K)$, and consider the following double Laplace transform

$$L(\lambda,\mu) := \int_0^{+\infty}\int_{-\infty}^{+\infty} \mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-rt}\big(Y_t - \mathrm{e}^{-k}\big)^+\big]\mathrm{e}^{-\mu t-\lambda k}\mathrm{d}k\mathrm{d}t,\ (\lambda,\mu) \in (0,+\infty).$$

Show that if τ is a random variable, independent of $B^{\mathbb{Q}}$, with exponential distribution[1] with parameter μ, then we have

$$L(\lambda,\mu) = \frac{1}{\mu\lambda(1+\lambda)}\mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-r\tau}Y_\tau^{1+\lambda}\big].$$

[1]We remind you that τ has exponential distribution with parameter μ is τ is a continuous random variable, valued in $\mathbb{R}$, with density

$$f_\tau(x) := \begin{cases} 0,\ \text{if } x \leq 0, \\ \mu\mathrm{e}^{-\mu x},\ \text{if } x > 0. \end{cases}$$

12) Check that for all $t \geq 0$

$$S_t^{1+\lambda} = S_0^{1+\lambda} \exp\bigg((1+\lambda)\Big(r + \frac{\sigma^2}{2}\lambda\Big)t\bigg)M_t, \text{ where } M_t := \exp\bigg((1+\lambda)\sigma B_t^{\mathbb{Q}} - \frac{1}{2}(1+\lambda)^2\sigma^2 t\bigg),\ t \geq 0.$$

Show also that the probability measure $\widehat{\mathbb{Q}}$, whose density with respect to $\mathbb{Q}$ on $\mathcal{F}_t$, is given for any $t \geq 0$ by M_t is well-defined, and explain why the following process is an $(\mathbb{F}, \widehat{\mathbb{Q}})-$Brownian motion

$$B_t^{\widehat{\mathbb{Q}}} := B_t^{\mathbb{Q}} - (1+\lambda)\sigma t,\ t \geq 0.$$

13) Show that $L(\lambda,\mu) = \frac{S_0^{1+\lambda}}{\lambda(1+\lambda)}\mathbb{E}^{\widehat{\mathbb{Q}}}\big[\int_0^{+\infty} \mathrm{e}^{-\beta t}\Psi_t^{1+\lambda}\mathrm{d}t\big]$, where

$$\Psi_t := \frac{Y_t}{S_t},\ t \geq 0,\ \beta := r + \mu - (1+\lambda)\Big(r + \frac{\sigma^2}{2}\lambda\Big).$$

14)($\star$) Use Itô's formula to deduce $\Psi_t = \int_0^t \big(1/T - \Psi_s(\lambda\sigma^2 + r)\big)\mathrm{d}s - \int_0^t \sigma\Psi_s \mathrm{d}B_s^{\widehat{\mathbb{Q}}}$. Deduce (informally) that there exists some function f such that

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\bigg[\int_t^{+\infty} \mathrm{e}^{-\beta(s-t)}\Psi_s^{1+\lambda}\mathrm{d}s\bigg|\mathcal{F}_t\bigg] = f(\Psi_t),\ t \geq 0.$$

Assuming that f is smooth, and using Itô's formula, deduce that f satisfies a linear ordinary differential equations that you will explicitly write. How can you use these results to price the Asian option?

Hint: there are very efficient numerical methods allowing to deduce the distribution of a random variable if the corresponding Laplace transform is known explicitly.